

Observers in spacetimes with spherical and axial symmetries

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Abstract

We introduce in the explicit form the tetrads of arbitrary observers in spacetimes with spherical and axial symmetries. The observers confined to the equatorial plane are parametrized by the pair of functions. We apply this description in the analysis of the null-geodesics in the observers' frames. The observers with the constant acceleration are distinguished.

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1 Introduction

Problem of the measurements in an observer frame is essential for any physical theory. In physical general relativistic system the measurements are observer-dependent so, it is important to compare their outcomes in different frames of reference. On the other hand the measurement has a physical meaning if it is given by the diffeomorphism invariant quantity. Thus one needs to identify such invariants. This problem has been solved long time ago by Élie Cartan [1] for the manifolds. However the problem of observables in general relativity have been reconsidered e.g. [2, 3, 4].

In this paper we will consider observers in space-times with fixed symmetry. We will establish the parametrization of the frames of reference of these observers by the pair of functions. In general the observer is determined by a time-like normalized (local) vector field on spacetime and provides a one-dimensional (timelike) foliation. Thus one defines the observer's frame as the manner of the splitting of the spacetime M on the time and space. Locally this splitting is always possible and means that the plane $T_x M$ tangent to M in the point x has the decomposition: $T_x M = \mathbf{R}u_{(0)} \oplus \Sigma_p$, where $u_{(0)}$ is a timelike vector and Σ_p is a vector space given by the three spacelike vectors $\{u_{(i)}\}$. These vectors constitute a local orthonormal basis: $\langle u_{(\alpha)}, u_{(\beta)} \rangle = \eta_{\alpha\beta}$ in the spacetime metric $\langle \cdot, \cdot \rangle$. Hence the all measures are performed with respect to this vector basis.

In the section 2 we will define the family of radial and transversal observers in the spherically symmetric spacetimes identifying relations between them. We

will distinguish the class of observers with the constant acceleration. They are generalizing the observers moving on the hyperbolic trajectories in the flat Minkowski spacetime.

In the section 3 we will consider the null geodesics in the frames of the observers with the constant acceleration. As the examples we will find relations of these null geodesics to the observers in the Schwarzschild and the Reissner–Nordström–de Sitter spacetimes.

In the section 4 we will generalize the results of the section 2 to the spacetimes with the axial symmetry and in the section 5 we will apply the obtained results to the Kerr spacetime. The section 6 is devoted to conclusions.

2 Observers in the spherically symmetric spacetimes

An arbitrary observer is determined by a velocity time-like vector field u

$$u^2 = 1. \quad (2.1)$$

This vector field gives the splitting of the space-time on the time and space. The temporal direction is given by $u_{(0)} \equiv u$ and the spatial directions are given by three space-like vector fields: $u_{(i)}$ and $i = 1, 2, 3$. Vector fields $(u_{(0)}, u_{(i)}) \equiv (u_{(\alpha)})$ (where $\alpha = 0, \dots, 3$) form the local tetrad:

$$u_{(\alpha)} \cdot u_{(\beta)} \equiv g_{\mu\nu} u_{(\alpha)}^\mu u_{(\beta)}^\nu = \eta_{\alpha\beta}, \quad (2.2)$$

where $(\eta_{\alpha\beta}) = \text{diag}(+1, -1, -1, -1)$. In a general case the vector field u has the four components: $(u^t, u^r, u^\theta, u^\phi)$ (in the spherical coordinate system (r, θ, ϕ)). The 10 equations (2.2) relate 16 components of the tetrad $(u_{(\alpha)}^\mu)$. Hence the tetrad of the observer is determined by the six independent parameters.

In a static spherically symmetric spacetime M with a metric $g_{\mu\nu}$:

$$ds^2 = g_0 dt^2 - g_1 dr^2 - g_2 d\theta^2 - g_3 d\phi^2, \quad (2.3)$$

where the components g_0 and g_1 depend only on the "radial" coordinate r and: $g_2 = r^2$, $g_3 = r^2 \sin^2 \theta$ we will consider a class of observers following planar trajectories. Without loss of generality we will assume that $\theta = \pi/2$ so the vector field u has the form:

$$u = \alpha \partial_t + \beta \partial_r + \gamma \partial_\phi \quad (2.4)$$

We can parametrize the vector field (2.4) by a pair of functions $(q(r), \chi(r))$ as follows:

$$u = \frac{1}{\sqrt{g_0}} \cosh q \partial_t + \frac{1}{\sqrt{g_1}} \sinh q \cos \chi \partial_r + \frac{1}{\sqrt{g_3}} \sinh q \sin \chi \partial_\phi. \quad (2.5)$$

From the equations (2.2) we get the other space-like vectors $u_{(i)}$:

$$u_{(1)} = \frac{1}{\sqrt{g_0}} \sinh q \partial_t + \frac{1}{\sqrt{g_1}} \cosh q \cos \chi \partial_r + \frac{1}{\sqrt{g_3}} \cosh q \sin \chi \partial_\phi, \quad (2.6)$$

$$u_{(2)} = \frac{1}{\sqrt{g_2}} \partial_\theta, \quad (2.7)$$

$$u_{(3)} = -\frac{\sin \chi}{\sqrt{g_1}} \partial_r + \frac{\cos \chi}{\sqrt{g_3}} \partial_\phi. \quad (2.8)$$

In this way we have obtained the tetrad given by (2.5-8). The (equatorial) trajectory of the observer (2.5) is given by an integral curve $\gamma = (t(\lambda), r(\lambda), \theta(\lambda) = \pi/2, \phi(\lambda))$ of u :

$$\frac{dt}{d\lambda} = \frac{1}{\sqrt{g_0}} \cosh q, \quad \frac{dr}{d\lambda} = \frac{1}{\sqrt{g_1}} \sinh q \cos \chi, \quad (2.9)$$

$$\frac{d\theta}{d\lambda} = 0, \quad \frac{d\phi}{d\lambda} = \frac{1}{\sqrt{g_3}} \sinh q \sin \chi, \quad (2.10)$$

where λ is a parameter of γ . An acceleration $a = a^\mu \partial_\mu$ of the observer (2.5) is given by:

$$a = \nabla_u u = u^\nu (\partial_\nu u^\mu + \Gamma_{\nu\rho}^\mu u^\rho) \partial_\mu. \quad (2.13)$$

The Christoffel symbols $\Gamma_{\nu\rho}^\mu$ for the metric (2.3) and the components of a are presented in the Appendix. Thus we get the relations ($a^\theta = 0$):

$$a^t = \frac{\sinh q \cos \chi}{g_0 \sqrt{g_1}} \frac{d}{dr} [\sqrt{g_0} \cosh q], \quad (2.14)$$

$$a^r = \frac{1}{2g_1} \frac{d}{dr} (\sinh^2 q \cos^2 \chi) + \frac{g_0'}{2g_1 g_0} \cosh^2 q - \frac{1}{r g_1} \sinh^2 q \sin^2 \chi, \quad (2.15)$$

$$a^\phi = \frac{\sinh q \cos \chi}{r^2 \sin \theta \sqrt{g_1}} \frac{d}{dr} (r \sinh q \sin \chi). \quad (2.16)$$

Moreover the acceleration is orthogonal to the velocity: $a \cdot u = 0$ (the dot means the scalar product with respect to the metric 2.3) Thus in the general case a is the linear combination of the vectors (2.6-8): $a = a_i u_{(i)}$, where $i = 1, 2, 3$.

There is the special class of the observers with the constant acceleration: $a \cdot a = \alpha^2 = \text{const}$. In the basis $u_{(i)}$ this condition has the form: $(a_1)^2 + (a_2)^2 + (a_3)^2 = \alpha^2$ and a_i are the constant number. Because $a^\theta = 0$ we get $a_2 = 0$ and the equations (2.14-16) take the form:

$$\cos \chi \frac{d}{dr} [\sqrt{g_0} \cosh q] = a_1 \sqrt{g_0 g_1}, \quad (2.17)$$

$$\begin{aligned} \frac{d}{dr} (\sinh^2 q \cos^2 \chi) + \frac{g_0'}{g_0} \cosh^2 q - \frac{2}{r} \sinh^2 q \sin^2 \chi = \\ 2\sqrt{g_1} (a_1 \cosh q \cos \chi - a_3 \sin \chi), \end{aligned} \quad (2.18)$$

$$\sinh q \cos \chi \frac{d}{dr} (r \sinh q \sin \chi) = \sqrt{g_1 g_3} (a_1 \cosh q \sin \chi + a_3 \cos \chi). \quad (2.19)$$

There is a special solution of these equations when $a_1 = a_2 = a_3 = 0$. It has the form:

$$\cosh q = E/\sqrt{g_0}, \quad (2.20)$$

$$\sinh q \cos \chi = \pm \sqrt{\frac{E^2}{g_0} - \frac{L^2 \sin^2 \theta}{r^2} - 1}, \quad (2.21)$$

$$\sinh q \sin \chi = \frac{L \sin \theta}{r}. \quad (2.22)$$

This solution corresponds to the time-like geodesic of the observer given by:

$$u = \frac{E}{g_0} \partial_t \pm \frac{1}{\sqrt{g_1}} \left[\sqrt{\frac{E^2}{g_0} - \frac{L^2 \sin^2 \theta}{r^2} - 1} \right] \partial_r + \frac{L}{r^2} \partial_\phi. \quad (2.23)$$

Next we consider two classes of observers: the *radial* observers $\chi = 0$ and the *transversal* observers, $\chi = \pi/2$. These two classes parametrized by the one function $q(r)$ only

2.1 The radial observer, $\chi = 0$

Let us first examine the radial observers. In this case the tetrad is:

$$u = \frac{1}{\sqrt{g_0}} \cosh q \partial_t + \frac{1}{\sqrt{g_1}} \sinh q \partial_r, \quad (2.24)$$

$$u_{(1)} = \frac{1}{\sqrt{g_0}} \sinh q \partial_t + \frac{1}{\sqrt{g_1}} \cosh q \partial_r, \quad (2.25)$$

$$u_{(2)} = \frac{1}{\sqrt{g_2}} \partial_\theta, \quad u_{(3)} = \frac{1}{\sqrt{g_3}} \partial_\phi. \quad (2.26)$$

The trajectory of the this observer is given by the equations:

$$\frac{dt}{d\lambda} = \frac{1}{\sqrt{g_0}} \cosh q, \quad \frac{dr}{d\lambda} = \frac{1}{\sqrt{g_1}} \sinh q, \quad (2.27)$$

$$\frac{d\theta}{d\lambda} = 0, \quad \frac{d\phi}{d\lambda} = 0, \quad (2.28)$$

where λ is a parameter of γ . For the observer with the constant acceleration the equations (2.17-19) have the solution:

$$\cosh q(r) = \frac{1}{\sqrt{g_0}} \left(E + \alpha \int^r \sqrt{g_0 g_1} dr \right) \quad (2.30)$$

$$\sinh q(r) = \pm \sqrt{\frac{1}{g_0} \left(E + \alpha \int^r \sqrt{g_0 g_1} dr \right)^2 - 1}, \quad (2.31)$$

where $a_1 = \alpha$ and $a_3 = 0$. The radial observer moves on the geodesic if $a^t = a^r = 0$. This condition leads to the relation: $\cosh q = E/\sqrt{g_0}$ with the vector field:

$$u = \frac{E}{g_0} \partial_t + \frac{1}{\sqrt{g_1}} \left[\frac{E^2}{g_0} - 1 \right]^{1/2} \partial_r. \quad (2.32)$$

2.2 The transversal observer, $\chi = \pi/2$

In this case the tetrad has basis vectors:

$$u = \frac{1}{\sqrt{g_0}} \cosh q \partial_t + \frac{1}{\sqrt{g_3}} \sinh q \partial_\phi, \quad (2.33)$$

$$u_{(1)} = \frac{1}{\sqrt{g_0}} \sinh q \partial_t + \frac{1}{\sqrt{g_3}} \cosh q \partial_\phi, \quad (2.34)$$

$$u_{(2)} = \frac{1}{\sqrt{g_2}} \partial_\theta, \quad u_{(3)} = -\frac{1}{\sqrt{g_1}} \partial_r. \quad (2.35)$$

The trajectory of such an observer is given by the equations:

$$\frac{dt}{d\lambda} = \frac{1}{\sqrt{g_0}} \cosh q, \quad \frac{d\phi}{d\lambda} = \frac{1}{\sqrt{g_3}} \sinh q, \quad (2.36)$$

$$\frac{d\theta}{d\lambda} = 0, \quad \frac{dr}{d\lambda} = 0, \quad (2.37)$$

where λ is a parameter of γ . For the observer with the constant acceleration α the equations (2.17-19) have the solution:

$$\cosh^2 q(r) = 2g_0 \frac{1 + ra_3 \sqrt{g_1}}{2g_0 - rg'_0} \geq 1, \quad (2.38)$$

$$\sinh^2 q(r) = 2g_0 \frac{1 + ra_3 \sqrt{g_1}}{2g_0 - rg'_0} - 1, \quad (2.39)$$

where $a_1 = 0$ and $(a_3)^2 = \alpha^2$. From the eq.(2.39) we obtain the allowed radius of the transversal observer:

$$2g_0 - rg'_0 > 0 \quad (2.40)$$

and:

$$2g_0 a_3 \sqrt{g_1} + g'_0 > 0. \quad (2.41)$$

In the case of the "free fall" $a_3 = 0$ comparing (2.38-39) and (2.21-23) one obtains:

$$rg_0 = \frac{1}{\sqrt{2}} \frac{L}{\sqrt{1 - 1/E^2}}. \quad (2.42)$$

The relations (2.40) and (2.41) for the Schwarzschild metric $g_0 = 1 - r_S/r$ yield the constraint on L and E :

$$2L^2 \sin^2 \theta > r_S^2 (1 - 1/E^2)$$

and the well known condition on the allowed radii of the circular orbits: $r > 3r_S/2$.

2.3 Relative velocity

According to [5] the relative velocity v between two observers determined by u and u' at the same point x is:

$$v = \frac{1}{g(u', u)} u' - u. \quad (2.43)$$

In our parametrization these observers are given by two pairs of functions (q, χ) and (q', χ') . Hence the scalar product $g(u', u)$ is equal to:

$$g(u', u) = \cosh q' \cosh q - \sinh q' \sinh q \cos(\chi' - \chi).$$

In the special case $\chi' = \chi$ we get the relative velocity:

$$v = \frac{\tanh(q' - q)}{\sqrt{g_0}} \sinh q \partial_t + \frac{\tanh(q' - q)}{\sqrt{g_1}} \cosh q \cos \chi \partial_r + \frac{\tanh(q' - q)}{\sqrt{g_3}} \cosh q \sin \chi \partial_\phi.$$

One can see that this velocity is the space-like vector such that:

$$v = u_{(1)} \tanh(q' - q) \quad (2.44)$$

and $u_{(1)}$ is given by eq. (2.6). For the radial observers ($\chi' = \chi = 0$) with the constant accelerations α and α' we obtain:

$$v^t = \frac{\sqrt{x^2 - 1}}{\sqrt{g_0}} \tanh \left[\ln \left(\frac{y + \sqrt{y^2 - 1}}{x + \sqrt{x^2 - 1}} \right) \right],$$

$$v^r = \frac{x}{\sqrt{g_1}} \tanh \left[\ln \left(\frac{y + \sqrt{y^2 - 1}}{x + \sqrt{x^2 - 1}} \right) \right],$$

where:

$$x = \frac{1}{\sqrt{g_0}} \left(E + \alpha \int^r \sqrt{g_0 g_1} dr \right)$$

and

$$y = \frac{1}{\sqrt{g_0}} \left(E' + \alpha' \int^r \sqrt{g_0 g_1} dr \right).$$

3 Null-geodesics in the observers frames with the constant acceleration

Let us now consider the null-geodesics that correspond to the rays in the geometric optics in the radial observer's frame. The null-geodesic is given by the null vector field $k = k^\mu \partial_\mu$ which is tangent to a curve γ parametrized by an affine parameter λ . Then the equation of γ in the coordinate system x^μ is obtained from the relations:

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) = 0, \quad (3.1)$$

where $L(x, \dot{x}) = \frac{1}{2} g_{\mu\nu}(x) k^\mu k^\nu = 0$ and $k^\nu \equiv \dot{x}^\nu$. For the metric (2.4) the eqs. (3.1) have the solutions:

$$g_0 k^0 = e, \quad g_3 k^3 = j, \quad (3.2)$$

where e and j can be interpreted as an energy and an angular momentum of a photon, respectively. The constraint $L = 0$ yields the relation:

$$\left(\frac{dr}{d\lambda}\right)^2 = (k^1)^2 = \frac{1}{g_1} \left(\frac{e^2}{g_0} - \frac{j^2}{r^2}\right) \quad (3.3)$$

The trajectories (3.1) are planar, so we fixed the coordinate $\theta = \pi/2$ which puts the constraint on a impact parameter $b = j/e$ of the photon: $r^2/g_0 > b^2$ for $g_0(r) > 0$. This equation can be rewritten as follows:

$$\left(\frac{dR}{j d\lambda}\right)^2 + V(R) = \frac{1}{b^2}, \quad (3.4)$$

where the new function $R = R(r)$ is given by the equation: $dR = \sqrt{g_0 g_1} dr$ and $V(R) = g_0(r(R))/r^2(R)$. This equation has the standard form of a particle motion in the potential $V(R)$. Thus the extremum of V is obtained from the relation $dV/dR = 0$ which leads to the equation:

$$\frac{1}{r^3 \sqrt{g_0(r) g_1(r)}} \left(r \frac{dg_0}{dr} - 2g_0(r) \right) = 0. \quad (3.5)$$

The solutions of this equation we call $r_e^{(p)}$:

$$\left. \frac{dV}{dR} \right|_{R=r_e^{(p)}} = 0, \quad (3.6)$$

where p numbers of the consecutive solutions of (3.5). We need to know the sign of $V''(R(r_e))$. It will depend on the form of the spherically symmetric metric which is the solution of the Einstein equations with an energy-momentum tensor $T_{\mu\nu}$. It is well known that if $T_{\mu\nu} = 0$, then are only: the Minkowski or the Schwarzschild-like metrics. Here we consider spherically symmetric metrics with the non vanishing energy-momentum tensor. Thus we can not apriori exclude the case where besides of the maximum of V there are minima of V . The maximum of V we call r_M and minimum r_m . Thus if $b^{-2} > V(R(r_M))$, then the whole spacetime is available for the trajectory of the photon (modulo horizons, if there are). If $b^{-2} < V(R(r_M))$, then the photon will hit the potential barrier and return to the infinity with $dr/d\lambda > 0$. In such a case there would arise a region between r_M and r_m where trapped photon trajectories would exist.

According to [6 (Box 25.7)] ψ is defined as an angle between propagation direction of a photon and the radial direction of the observer. In the case of the Schwarzschild metric a cone of avoidance with an apex in an event x_0^μ is defined [7 (ch. 20)] by the null critical geodesics that passes x_0^μ . An angle ψ is an half-angle of the cone.

In the observer frame with the tetrad $(u_{(\alpha)}^\mu)$ the tangent vector field $k = k^\mu \partial_\mu$ to γ has the components:

$$k_{(\alpha)} = g_{\mu\nu} k^\mu u_{(\alpha)}^\nu. \quad (3.7)$$

Hence the four-vector k is: $k = k_{(0)} u + k_{(1)} u_{(1)} + k_{(3)} u_{(3)}$ and $(k_{(0)})^2 = (k_{(1)})^2 + (k_{(3)})^2$. Thus ψ is the angle between the spacelike vector $\mathbf{k} = k_{(1)} u_{(1)} + k_{(3)} u_{(3)}$

and the axis $u_{(3)}$ in the frame of the observer is expressed in arbitrary metric as follows:

$$\tan \psi = \frac{k_{(1)}}{k_{(3)}}. \quad (3.8)$$

In the spherically symmetric space-time with the diagonal metric and a radially moving observer we get:

$$k_{(0)} = \frac{e}{\sqrt{g_0}} \cosh q \pm \sqrt{\frac{e^2}{g_0} - \frac{j^2}{g_3}} \sinh q, \quad (3.9)$$

$$k_{(1)} = \frac{e}{\sqrt{g_0}} \sinh q \pm \sqrt{\frac{e^2}{g_0} - \frac{j^2}{g_3}} \cosh q, \quad (3.10)$$

$$k_{(3)} = \frac{j}{\sqrt{g_3}}. \quad (3.11)$$

The $-$ sign corresponds to photons coming from the infinity to the observer and $+$ sign corresponds to the photons that can reach the observer provided their impact parameter satisfies: $b^{-2} < V(R(r_M))$. It leads to the relation:

$$\tan \psi_{\pm} = \frac{1}{b} \sqrt{\frac{g_3}{g_0}} \left[\sinh q \pm \sqrt{1 - b^2 \frac{g_0}{g_3}} \cosh q \right], \quad (3.12)$$

One can rewrite the last formula as follows:

$$\tan \psi_{\pm} = \sinh (q \pm w), \quad (3.12a)$$

where $\cosh w = \sqrt{g_3/(b^2 g_0)}$. For the radial observer with the constant acceleration (2.30) and $\theta = \pi/2$ ($g_3 = r^2$) we obtain:

$$q \pm w = \ln \left[|b| \frac{s + \sqrt{s^2 - g_0}}{r \mp \sqrt{r^2 - b^2 g_0}} \right], \quad (3.13)$$

where $s = E + \alpha \int^r \sqrt{g_0 g_1} dr$. Hence $q + w \rightarrow \infty$ if $g_0 \rightarrow 0$ what means that $\psi \rightarrow \pi/2$ on the horizon.

3.1 Reissner–Nordström–de Sitter spacetime

This spacetime is a spherically symmetric solution of the Einstein equations [8] described by the three parameters: a mass m , an electric charge e and a cosmological constant Λ ; it generalizes the Schwarzschild spacetime. It represents a black hole, that generally possess three horizons according to the nonzero values of m , e and Λ . These horizons correspond to outer (event) and inner (Cauchy) black hole horizons and the cosmological horizon. It may therefore be interpreted as the space-time of a charged black hole in a de Sitter or anti-de Sitter spacetimes. The metric is:

$$ds^2 = g_0(r) dt^2 - \frac{1}{g_0(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.14)$$

where:

$$g_0(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3} r^2.$$

In order to simplify calculations we consider the uncharged black hole $e = 0$ in the de Sitter $\Lambda > 0$ or anti-deSitter $\Lambda < 0$ spacetimes. Thus the horizons are given by the solution of the cubic equation:

$$r^3 - \frac{3}{\Lambda}r + \frac{6m}{\Lambda} = 0$$

In the de Sitter case and $m > 0$ one gets that the number of real roots depends on the sign of the discriminant $D = -1/\Lambda^3 + 9m^2/\Lambda^2$. If $D > 0$ (it means that: $9m^2\Lambda > 1$), then there is no real positive root. If $D < 0$ (it means that: $0 < 9m^2\Lambda < 1$), then there are two real positive roots $r_+ > r_- > 0$:

$$r_{\pm} = \frac{2}{\sqrt{\Lambda}} \cos \left[\frac{\pi}{3} \mp \frac{1}{3} \arccos \left(3m\sqrt{\Lambda} \right) \right] \quad (3.15)$$

and one negative root $r_0 < 0$. Moreover the sum of these roots vanishes: $r_+ + r_- + r_0 = 0$. Hence one gets:

$$r_0 = -\frac{2}{\sqrt{\Lambda}} \cos \left[\frac{1}{3} \arccos \left(3m\sqrt{\Lambda} \right) \right].$$

These positive roots determine the cosmological horizon (r_+) and the event horizon (r_-). Hence the metric component g_0 takes the form:

$$g_0(r) = \frac{\Lambda}{3r} (r_+ - r)(r - r_-)(r - r_0)$$

and the metric is stationary for $r \in (r_-, r_+)$. One can notice that for $9m^2\Lambda = 1$ the cosmological and event horizons coincide: $r_+ = r_- = 3m = 1/\sqrt{\Lambda}$ and:

$$g_0(r) = -\frac{\Lambda}{3r} (r - 3m)^2 (r + 6m).$$

In this case the coordinate r becomes temporal-like while t becomes spatial-like. If $m = 0$, then the metric is the de Sitter metric in the static coordinates and the cosmological horizon is given by: $r_c = \sqrt{3/\Lambda}$.

In the anti-de Sitter case the equation $g_0(r)$ has the one real root:

$$r_h = \frac{2}{\sqrt{-\Lambda}} \sinh \left[\frac{1}{3} \sinh^{-1} \left(3m\sqrt{-\Lambda} \right) \right], \quad (3.16)$$

where $\sinh^{-1}(y)$ is the inverse of the $\sinh x = y$.

The angle ψ for $\Lambda > 0$ is given by:

$$\tan \psi = \frac{r}{b} \sqrt{\frac{3r}{\Lambda f(r)}} \left[\sinh q \pm \sqrt{1 - b^2 \frac{\Lambda f(r)}{3r^3}} \cosh q \right], \quad (3.17)$$

where: $f(r) = (r_+ - r)(r - r_-)(r + r_0) = -r^3 + 3r/\Lambda - 6m/\Lambda$.

In the case of the Schwarzschild metric the eq. (3.17) is:

$$\tan \psi = \frac{1}{\tilde{b}\sqrt{y-1}} \left[\frac{1}{\sqrt{y}} \sinh q \pm \sqrt{y^3 - \tilde{b}^2 y + \tilde{b}^2} \cosh q \right], \quad (3.18)$$

where we introduced the dimensionless variables $y = r/r_S > 1$ and $\tilde{b} = b/r_S$. Hence the psi angle is function of two parameters: q and \tilde{b} . For the observer with the constant acceleration eqs.(2.30-31) we obtain the formula:

$$\tan \psi = \frac{\sqrt{y}}{\tilde{b}(y-1)} \left[\frac{1}{y} \sqrt{y(E - r_s \alpha + y r_s \alpha)^2 + 1 - y} \pm \sqrt{y^3 - \tilde{b}^2 y + \tilde{b}^2 (E - r_s \alpha + y r_s \alpha)} \right]. \quad (3.19)$$

For the geodesic radial observer ($\alpha = 0$) the above formula reads:

$$\tan \psi = \frac{\sqrt{y}}{\tilde{b}(y-1)} \left[\frac{1}{y} \sqrt{y(E^2 - 1) + 1} \pm E \sqrt{y^3 - \tilde{b}^2 y + \tilde{b}^2} \right]. \quad (3.20)$$

For the static observer ($\sinh q = 0$) the formula (3.18) gives the relation:

$$\cot^2 \psi(y; \tilde{b}) = \tilde{b}^2 \frac{y-1}{y^3 - \tilde{b}^2 y + \tilde{b}^2}. \quad (3.21)$$

The other form of (3.19) is:

$$0 \leq \sin^2 \psi(y; \tilde{b}) = \tilde{b}^2 \frac{y-1}{y^3} \leq 1. \quad (3.22)$$

From the last equation we get the constraint relating \tilde{b} and y :

$$y^3 - \tilde{b}^2 y + \tilde{b}^2 \geq 0. \quad (3.23)$$

4 Observers in the spacetime with the axial symmetry

In this section we will consider a spacetime with the axial symmetry having two Killing vectors $V_{(t)} = \partial_t$ and $V_{(\phi)} = \partial_\phi$. A metric $g_{\mu\nu}$ has the non-diagonal form:

$$ds^2 = g_0 dt^2 - g_1 dr^2 - g_2 d\theta^2 - g_3 d\phi^2 - 2h dt d\phi \quad (4.1)$$

and the coefficients g_a and h are the functions of r and θ . An observer in the space-time (4.1) for $\theta = \text{const}$ is given by a time-like vector field u :

$$u \equiv u_{(0)} = \alpha \partial_t + \beta \partial_r + \gamma \partial_\phi. \quad (4.2)$$

The condition $u^2 = 1$ takes the form: $\alpha^2 D/g_3 - \beta^2 g_1 - (\alpha h + \gamma g_3)^2/g_3 = 1$, where: $D = h^2 + g_0 g_3$. Also in this case we can parametrize the vector field u by a pair of functions: $(q(r), \chi(r))$:

$$u = \sqrt{\frac{g_3}{D}} \cosh q \partial_t + \frac{1}{\sqrt{g_1}} \sinh q \cos \chi \partial_r + \frac{1}{\sqrt{g_3}} \left(\sinh q \sin \chi - \frac{h}{\sqrt{D}} \cosh q \right) \partial_\phi. \quad (4.3)$$

The other three basis vectors $u_{(i)}$ are:

$$u_{(1)} = \sqrt{\frac{g_3}{D}} \sinh q \partial_t + \frac{1}{\sqrt{g_1}} \cosh q \cos \chi \partial_r + \frac{1}{\sqrt{g_3}} \left(\cosh q \sin \chi - \frac{h}{\sqrt{D}} \sinh q \right) \partial_\phi, \quad (4.4)$$

$$u_{(2)} = \frac{1}{\sqrt{g_2}} \partial_\theta, \quad (4.5)$$

$$u_{(3)} = -\frac{\sin \chi}{\sqrt{g_1}} \partial_r + \frac{\cos \chi}{\sqrt{g_3}} \partial_\phi. \quad (4.6)$$

In this way we determine the observer whose frame of reference is given by (4.3-6). In the case $h = 0$ we get (2.5-8). The integral curve for (4.3) is given by the equations:

$$\frac{dt}{d\lambda} = \sqrt{\frac{g_3}{D}} \cosh q, \quad (4.7)$$

$$\frac{dr}{d\lambda} = \frac{1}{\sqrt{g_1}} \sinh q \cos \chi, \quad \frac{d\theta}{d\lambda} = 0, \quad (4.8)$$

$$\frac{d\phi}{d\lambda} = \frac{1}{\sqrt{g_3}} \left(\sinh q \sin \chi - \frac{h}{\sqrt{D}} \cosh q \right). \quad (4.9)$$

From the equations (4.7-9) we can see that the static observer ($dr/d\lambda = d\theta/d\lambda = d\phi/d\lambda = 0$) is realized by the functions: $\chi = \pi/2$ and $\tanh q = h/\sqrt{D}$ so $\cosh q = \sqrt{D/(g_0 g_3)}$ and $\sinh q = h/\sqrt{g_0 g_3}$. The radial trajectories ($\phi = \text{const}$) are given by the conditions

$$\tanh q \sin \chi = h/\sqrt{D} \quad \text{and} \quad \chi \neq 0. \quad (4.10)$$

Next we consider a null-geodesic with a tangent vector $k = k^\mu \partial_\mu$ in this space-time. The Euler-Lagrange equations gives the conserved quantities e and j :

$$g_0 k^0 - h k^3 = e, \quad (4.11)$$

$$h k^0 + g_3 k^3 = j. \quad (4.12)$$

It is easy solve the above equations so:

$$k^0 = \frac{D_0(e, j)}{D}, \quad k^3 = \frac{D_3(e, j)}{D}, \quad (4.13)$$

where $D_0(e, j) = hj + g_3 e$ and $D_3(e, j) = -he + g_0 j$. There is the constraint $k \cdot k = 0$ that provides the relation between the k^1 and k^2 :

$$g_1 (k^1)^2 + g_2 (k^2)^2 = \frac{W(r, \theta; e, j)}{g_3 D}, \quad (4.14)$$

where: $W(r, \theta; e, j) \equiv D_0^2 - j^2 D$. For the Kerr metric the constraint (4.14) can be separated. It means that there are two functions $R(r)$ and $\Theta(\theta)$ such that: $k^1/\sqrt{R(r)} = \pm k^2/\sqrt{\Theta(\theta)}$ (see Section 5). We can solve the constraint (4.14) introducing a new function σ of r and θ such that:

$$k^1 = \sqrt{\frac{W}{g_1 g_3 D}} \cos \sigma(r, \theta), \quad k^2 = \sqrt{\frac{W}{g_2 g_3 D}} \sin \sigma(r, \theta). \quad (4.15)$$

The function σ has the interpretation of an angle in the plane (k^1, k^2) .

If the null vector k has the second component k^2 equal to zero, then ψ in the observer's frame is given by the eq. (3.8). One can say that the null geodesic

given by k is confined to the equatorial plane $\theta = \pi/2$ and $k^1 = \sqrt{W/(g_1 g_3 D)}$. Then the equation (3.7) gives the components of k in the observer's frame:

$$k_{(1)} = \frac{D_0}{\sqrt{g_3 D}} \sinh q - \frac{1}{\sqrt{g_3}} \left(\sqrt{W/D} \cos \chi + j \sin \chi \right) \cosh q, \quad (4.16)$$

$$k_{(2)} = 0, \quad k_{(3)} = \sqrt{\frac{W}{g_3 D}} \sin \chi - \frac{j}{\sqrt{g_3}} \cos \chi. \quad (4.17)$$

Hence we get:

$$\tan \psi(r; q, \chi) = \frac{1}{\sqrt{D}} \frac{D_0 \sinh q - \left(\sqrt{\frac{D_0^2}{D} - j^2} \cos \chi + j \sin \chi \right) \sqrt{D} \cosh q}{\sqrt{\frac{D_0^2}{D} - j^2} \sin \chi - j \cos \chi}, \quad (4.18)$$

where $D_0 = D_0(e, j)$ depends on the parameters e and j describing the null-geodesic.

For the radial trajectory of the observer in the equatorial plane ($\tanh q \sin \chi = h/\sqrt{D}$) and for $\chi = \pi/4$ there is the relation: $\sqrt{D} \sinh q = h\sqrt{2} \cosh q$ so the observer velocity vector u takes the form:

$$u = \sqrt{\frac{g_3}{D}} \cosh q \partial_t + \frac{h}{\sqrt{g_1 D}} \cosh q \partial_r. \quad (4.19)$$

Moreover from the condition $u^2 = +1$ we get: $\cosh q = D^{1/2}(g_0 g_3 - h^2)^{-1/2}$. Thus the formula (4.18) is:

$$\tan \psi = \frac{1}{\sqrt{D(g_0 g_3 - h^2)}} \times \frac{2D_0 h - \left(\sqrt{\frac{D_0^2}{D} - j^2} + j \right) D}{\sqrt{\frac{D_0^2}{D} - j^2} - j}. \quad (4.20)$$

5 Kerr metric

In this section we will apply the results of the section 4 to the Kerr metric. This metric in terms of the spheroidal-like coordinates of Boyer and Lindquist (1967) has the form:

$$\begin{aligned} ds^2 = & \left(1 - \frac{2mr}{\Sigma} \right) dt^2 + 4 \frac{mar \sin^2 \theta}{\Sigma} dt d\phi - \frac{\Sigma}{\Delta} dr^2 \\ & - \Sigma d\theta^2 - \left(r^2 + a^2 + 2 \frac{mra^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2. \end{aligned} \quad (5.1)$$

where $\Delta(r) = r^2 - 2mr + a^2$ and $\Sigma(r, \theta) = r^2 + a^2 \cos^2 \theta$. The variables (t, r, θ, ϕ) belong to the following intervals: $t \in R^1$, $r \in (0, r_-) \cup (r_-, r_+) \cup (r_+, +\infty)$, $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ where: $r_{\pm} = m \pm \sqrt{m^2 - a^2}$.

In order to obtain the angle ψ (4.18) we have to know the component k^1 and $D_0(e, j)$ of the null vector field k and the function q and χ defining the observer. The components k^μ of the null geodesic vector field $k = k^\mu \partial_\mu$ are well-known (e.g [9]) and have the form:

$$k^0 = \frac{1}{\Sigma} \left[a(j - ae \sin^2 \theta) + \frac{r^2 + a^2}{\Delta} P \right],$$

$$k^1 = \frac{\sqrt{R}}{\Sigma}, \quad k^2 = \frac{\sqrt{\Theta}}{\Sigma},$$

$$k^3 = -\frac{1}{\Sigma} \left[\left(ae - \frac{j}{\sin^2 \theta} \right) + \frac{a}{\Delta} P \right],$$

where:

$$P(r) = e(r^2 + a^2) - ja,$$

$$R(r) = P^2 - \left[C + (j - ae)^2 \right] \Delta,$$

$$\Theta(\theta) = C + a^2 \left(e^2 - \frac{j^2}{a^2 \sin^2 \theta} \right) \cos^2 \theta$$

and C is the separation constant in the equation (4.14). These components are the solutions of the general algebraic equations (4.13-14) for the axial symmetric spacetime. Hence we get:

$$D = \Delta(r) \sin^2 \theta, \quad (5.2)$$

$$D_0 = [e(r^2 + a^2) \Sigma - 2mra(j - a \sin^2 \theta)] \frac{\sin^2 \theta}{\Sigma}, \quad (5.3)$$

$$D_3 = j \left(1 - \frac{2mr}{\Sigma} \right) + 2e \frac{mar \sin^2 \theta}{\Sigma}. \quad (5.4)$$

We want to find the psi angle given by (4.18) so we only consider the null vectors in the equatorial plane $\theta = \pi/2$. Thus $\Sigma(r, \pi/2) = r^2$, $g_3(\pi/2) = r^2 + a^2 + 2ma^2/r$, $D(\pi/2) = \Delta(r)$ and:

$$D_0(\pi/2) = [e(r^2 + a^2)r - 2ma(j - a)] \frac{1}{r}.$$

We obtain then

$$\tan \psi(r; q, \chi) = \frac{D_0 \sinh q - \left(\sqrt{D_0^2 - j^2 \Delta} \cos \chi + j \sqrt{\Delta} \sin \chi \right) \cosh q}{\sqrt{D_0^2 - j^2 \Delta} \sin \chi - j \sqrt{\Delta} \cos \chi} \quad (5.5)$$

For the radial motion and $\chi = \pi/4$ we get from (4.19) :

$$\tan \psi = \frac{1}{\sqrt{\Delta(g_0 g_3 - h^2)}} \times \frac{2D_0 h - \left(\sqrt{\frac{D_0^2}{\Delta} - j^2} + j \right) \Delta}{\sqrt{\frac{D_0^2}{\Delta} - j^2} - j}, \quad (5.6)$$

where $h(r) = 2ma/r$ and $g_0 = 1 - 2m/r$. Thus the angle ψ depends on the two parameters of the null-geodesic: e, j .

In the case of the geodesic observer in the equatorial plane ($d\theta/d\lambda = 0$) the equations (4.7-9) are known in the explicit form:

$$\dot{t} \equiv \frac{dt}{d\lambda} = \frac{1}{\Delta} \left[\left(r^2 + a^2 + \frac{2ma^2}{r} \right) E - \frac{2ma}{r} J \right],$$

$$\dot{r}^2 \equiv \left(\frac{dr}{d\lambda} \right)^2 = \frac{1}{r^2} \left[-\Delta + r^2 E^2 + \frac{2m}{r} (J - aE)^2 - (J^2 - a^2 E^2) \right],$$

$$\dot{\phi} \equiv \frac{d\phi}{d\lambda} = \frac{1}{\Delta} \left[\left(1 - \frac{2m}{r} \right) J + \frac{2ma}{r} E \right],$$

where E and J are an energy and an angular momentum of the observer, respectively. From these equations one gets the functions q and χ describing the observer. Next inserting functions q and χ into the eq. (5.5) we get the following formula for the angle ψ :

$$\tan \psi = \frac{1}{\sqrt{\Delta}} \frac{D_0 \left(\dot{t}^2 g_3 - \Delta \right) - \sqrt{\Delta} \left(\dot{r} \sqrt{g_1} \sqrt{D_0^2 - j^2 \Delta} + j \sqrt{\dot{t}^2 g_3 - \Delta \left(1 + \dot{r}^2 g_1 \right)} \right)}{\sqrt{D_0^2 - j^2 \Delta} \sqrt{\dot{t}^2 g_3 - \Delta \left(1 + \dot{r}^2 g_1 \right)} - j \dot{r} \sqrt{g_1}},$$

where $g_1 = (1 - 2m/r + a^2/r^2)^{-1}$.

For the slow rotating Kerr black hole $a^2/r^2 \ll 1$ the metric takes the form:

$$ds^2 = \left(1 - \frac{2m}{r} \right) dt^2 + 4 \frac{ma \sin^2 \theta}{r} dt d\phi - \left(1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. In this case there is only one horizon: $r_h = 2m$. In the eq. (5.5) survived only terms linear in a . Thus ψ is given by:

$$\tan \psi(r; q, \chi) = \frac{D_0 \sinh q - \left(\sqrt{F} \cos \chi + j \sqrt{\Delta} \sin \chi \right) \cosh q}{\sqrt{F} \sin \chi - j \sqrt{\Delta} \cos \chi},$$

where $F = e^2 r^4 - j^2 r^2 + 2mrj(j - 2ae)$, $D_0 = er^2 - 2maj/r$ and $\Delta = r^2 - 2mr$.

In the general case we can obtain the pair (q, χ) and inserting them into the general formula (4.18) one obtains $\tan \psi$. As one can see this result will be highly complicated and crucially depending on the three null geodesic parameters (e, j, C) and three observer's parameters (E, J, K) . Here parameter K is the separation constant of the observer's time-like geodesic.

6 Conclusions

We have obtained the family of observers parametrized by the pair of functions (q, χ) . This parametrization was applied to determine the angle ψ in the spacetimes with spherical and axial symmetries. The angle ψ depends on the parameters describing the null geodesic of the photon (e, j) and parameters (q, χ) describing an observer. In the case of the spherical symmetry we get the family of the observers with the constant acceleration. For the radial and transversal observers we obtained the functions q in the explicit form (2.30) and (2.39) respectively. For the radial observer with the constant acceleration and the Schwarzschild metric we obtained the equation (3.11) for the the psi angle.

We extended our considerations to the more realistic spacetimes with the axial symmetry in the sections 4 and 5. In the case of the Kerr metric we obtained the compact expression for ψ in the case of the radial motion.

The obtained parametrization can be used in other applications.

7 Appendix

The non-vanishing Christoffel symbols for the metric (2.4) are:

$$\begin{aligned}\Gamma_{tr}^t &= \frac{g'_0}{2g_0}, \\ \Gamma_{tt}^r &= \frac{g'_0}{2g_1}, \quad \Gamma_{rr}^r = \frac{g'_1}{2g_1}, \quad \Gamma_{\theta\theta}^r = -\frac{r}{g_1}, \quad \Gamma_{\phi\phi}^r = -\frac{r \sin^2 \theta}{g_1}, \\ \Gamma_{r\theta}^\theta &= \frac{1}{r}, \quad \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta, \\ \Gamma_{r\phi}^\phi &= \frac{1}{r}, \quad \Gamma_{\theta\phi}^\phi = \cot \theta.\end{aligned}$$

The acceleration a of the vector field (2.5) for the metric (2.4) is:

$$\begin{aligned}a^t &= 2\Gamma_{tr}^t u^r u^t + u^r \partial_r u^t, \\ a^r &= \Gamma_{tt}^r (u^t)^2 + \Gamma_{rr}^r (u^r)^2 + \Gamma_{\phi\phi}^r (u^\phi)^2 + u^r \partial_r u^r, \\ a^\phi &= 2\Gamma_{r\phi}^\phi u^r u^\phi + u^r \partial_r u^\phi, \quad a^\theta = 0,\end{aligned}$$

where $u^t = \cosh q / \sqrt{g_0}$, $u^r = \sinh q \cos \chi / \sqrt{g_1}$ and $u^\phi = \sinh q \sin \chi / \sqrt{g_3}$.

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